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# CLOSE-TO-CONVEXITY, STARLIKENESS AND CONVEXITY

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**Abstract**

The main object of the present paper is to derive several sufficient conditions for close-to-convexity, starlikeness and convexity of certain (normalized) analytic functions. Relevant connections of some of the results obtained in this paper with those in earlier works are also provided.

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## 1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions $f$ *normalized* by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathcal{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Also let $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$, and $\mathcal{C}(\alpha)$ denote the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, *starlike*, *convex close-to-convex of order* $\alpha$ in $\mathcal{U}$ $(0 \leqq \alpha)$ . Thus we have (see, for details, Duren [1] and Goodman [2]; see also Srivastava and Owa [6])

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathcal{U};\, 0 \leqq \alpha < 1) \right\}, \tag{1.2}$$

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathcal{U};\, 0 \leqq \alpha < 1) \right\}, \tag{1.3}$$

and

$$\mathcal{C}(\alpha) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(\frac{f'(z)}{g'(z)}\right) > \alpha \quad (z \in \mathcal{U};\, 0 \leqq \alpha < 1;\, g \in \mathcal{K}) \right\}, \tag{1.4}$$

where, for convenience,

$$\mathcal{S}^* := \mathcal{S}^*(0), \quad \mathcal{K} := \mathcal{K}(0), \quad \text{and} \quad \mathcal{C} := \mathcal{C}(0). \tag{1.5}$$

Next, with a view to recalling the principle of subordination betwen analytic functions, let the functions $f$ and $g$ be analytic in $\mathcal{U}$. Then we say that the function $f$ is *subordinate* to $g$ if there exists a function $h$, analytic in $\mathcal{U}$, with

$$h(0) = 0 \quad \text{and} \quad |h(z)| < 1 \quad (z \in \mathcal{U}), \tag{1.6}$$

such that

$$f(z) = g(h(z)) \quad (z \in \mathcal{U}). \tag{1.7}$$

We denote this subordination by

$$f(z) \prec g(z). \tag{1.8}$$

In particular, if the function $g$ is *univalent* in $\mathcal{U}$, the subordination (1.8) is equivalent to (*cf.* [1, p. 190])

$$f(0) = g(0) \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U}). \tag{1.9}$$

Recently, R. Singh and S. Singh [5] proved several interesting results involving univalence and starlikeness of functions $f \in \mathcal{A}$. In our attempt here to generalize these results of Singh and Singh [5], we are led naturally to several sufficient conditions for close-to-convexity, starlikeness, and convexity of functions $f \in \mathcal{A}$.

The following lemma (popularly known as *Jack's lemma*) will be required in our present investigation.

**Lemma 1** (*cf.* Jack [3]; see also Miller and Mocanu [4]). *Let the (non-constant) function $w(z)$ be analytic in $\mathcal{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathcal{U}$, then*

$$z_0\, w'(z_0) = c\, w(z_0),$$

*where $c$ is a real number and $c \geqq 1$.*

## 2. Sufficient Conditions for Close-to-Convexity

Our first result (Theorem 1 below) provides a sufficient condition for close-to-convexity of functions $f \in \mathcal{A}$.

**Theorem 1.** *Let the function $f \in \mathcal{A}$ satisfy the inequality:*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{1+3\alpha}{2(1+\alpha)} \quad (z \in \mathcal{U};\ 0 \leqq \alpha < 1). \tag{2.1}$$

*Then*

$$\Re\{f'(z)\} > \frac{1-\alpha}{2} \quad (z \in \mathcal{U};\ 0 \leqq \alpha < 1) \tag{2.2}$$

*or, equivalently,*

$$f \in \mathcal{C}\left(\frac{1-\alpha}{2}\right) \quad (0 \leqq \alpha < 1). \tag{2.3}$$

**Proof.** We begin by defining a function $w$ by

$$f'(z) = \frac{1+\alpha w(z)}{1+w(z)} \quad (w(z) \neq -1;\ z \in \mathcal{U};\ 0 \leqq \alpha < 1). \tag{2.4}$$

Then, clearly, $w$ is analytic in $\mathcal{U}$ *with* $w(0) = 0$. We also find from (2.4) that

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{\alpha z w'(z)}{1+\alpha w(z)} - \frac{zw'(z)}{1+w(z)} \quad (z \in \mathcal{U}). \tag{2.5}$$

Suppose now that there exists a point $z_0 \in \mathcal{U}$ such that

$$|w(z_0)| = 1 \quad \text{and} \quad |w(z)| < 1 \quad \text{when} \quad |z| < |z_0|. \tag{2.6}$$

Then, by applying Lemma 1, we have

$$z_0\, w'(z_0) = c\, w(z_0) \quad \left(c \geqq 1;\ w(z_0) = e^{i\theta};\ \theta \in \mathbb{R}\right). \tag{2.7}$$

Thus we find from (2.5) and (2.7) that

$$\begin{aligned}\Re\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right) &= 1+\Re\left(\frac{c\alpha\, e^{i\theta}}{1+\alpha e^{i\theta}}\right)-\Re\left(\frac{ce^{i\theta}}{1+e^{i\theta}}\right) \\ &= 1+\frac{c\alpha\,(\alpha+\cos\theta)}{1+\alpha^2+2\alpha\cos\theta}-\frac{c}{2} \\ &\leqq \frac{1+3\alpha}{2(1+\alpha)} \quad (z_0\in\mathcal{U};\, 0\leqq\alpha<1),\end{aligned}$$

which obviously contradicts our hypothesis (2.1). It follows that

$$|w(z)|<1 \quad (z\in\mathcal{U}),$$

that is, that

$$\left|\frac{1-f'(z)}{f'(z)+\alpha}\right|<1 \quad (z\in\mathcal{U};\, 0\leqq\alpha<1). \tag{2.8}$$

This evidently completes the proof of Theorem 1.

**Theorem 2.** *If the function $f\in\mathcal{A}$ satisfies the inequality:*

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)<\frac{3+2\alpha}{2+\alpha} \quad (z\in\mathcal{U};\, 0\leqq\alpha<1), \tag{2.9}$$

*then*

$$|f'(z)-1|<1+\alpha \quad (z\in\mathcal{U};\, 0\leqq\alpha<1). \tag{2.10}$$

**Proof.** Our proof of Theorem 2, also based upon Lemma 1, is much akin to that of Theorem 1. Indeed, in place of the definition (2.4), here we let the function $w$ be given by

$$f'(z)=(1+\alpha)w(z)+1 \quad (z\in\mathcal{U};\, 0\leqq\alpha<1). \tag{2.11}$$

The details may be omitted.

**Remark 1.** Since the inequality (2.10) implies that

$$\Re\{f'(z)\}>-\alpha \quad (z\in\mathcal{U};\, 0\leqq\alpha<1), \tag{2.12}$$

by setting $\alpha=0$ in Theorem 2, we readily obtain

**Corollary 1** (Singh and Singh [5, p. 311, Corollary 2]). *If the function $f\in\mathcal{A}$ satisfies the inequality:*

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)<\frac{3}{2} \quad (z\in\mathcal{U}), \tag{2.13}$$

*then*

$$|f'(z)-1|<1 \quad (z\in\mathcal{U}), \tag{2.14}$$

*that is, $f\in\mathcal{C}$.*

Next we prove

**Theorem 3.** *If the function $f \in \mathcal{A}$ satisfies the inequality:*

$$|f'(z) - 1|^{\beta} \, |zf''(z)|^{\gamma} < \frac{(1-\alpha)^{\beta+\gamma}}{2^{\beta+2\gamma}} \quad (z \in \mathcal{U};\ 0 \leqq \alpha < 1;\ \beta, \gamma \geqq 0), \tag{2.15}$$

*then*

$$\Re\{f'(z)\} > \frac{1+\alpha}{2} \quad (z \in \mathcal{U};\ 0 \leqq \alpha < 1). \tag{2.16}$$

**Proof.** We define the function $w$ by

$$f'(z) = \frac{1+\alpha w(z)}{1+w(z)} \quad (w(z) \neq -1;\ z \in \mathcal{U};\ 0 \leqq \alpha < 1). \tag{2.17}$$

Then, clearly, $w$ is analytic in $\mathcal{U}$ *with* $w(0) = 0$. We also find from (2.17) that

$$|f'(z) - 1|^{\beta} \, |zf''(z)|^{\gamma} = \frac{(1-\alpha)^{\beta+\gamma} |w(z)|^{\beta} \, |zw'(z)|^{\gamma}}{|1+w(z)|^{\beta+2\gamma}} \quad (z \in \mathcal{U}). \tag{2.18}$$

Supposing now that there exists a point $z_0 \in \mathcal{U}$ such that

$$|w(z_0)| = 1 \quad \text{and} \quad |w(z)| < 1 \quad \text{when} \quad |z| < |z_0|,$$

if we apply Lemma 1 just as we did in the proof of Theorem 1, we shall obtain

$$\begin{aligned} |f'(z_0) - 1|^{\beta} \, |z_0 f''(z_0)|^{\gamma} &= \frac{(1-\alpha)^{\beta+\gamma} c^{\gamma}}{|1+e^{i\theta}|^{\beta+2\gamma}} \\ &\geqq \frac{(1-\alpha)^{\beta+\gamma}}{2^{\beta+2\gamma}} \quad (z_0 \in \mathcal{U};\ 0 \leqq \alpha < 1), \end{aligned}$$

which obviously contradicts our hypothesis (2.15). Thus we have

$$|w(z)| < 1 \quad (z \in \mathcal{U}),$$

which implies that

$$\left| \frac{f'(z) - 1}{f'(z) - \alpha} \right| < 1 \quad (z \in \mathcal{U};\ 0 \leqq \alpha < 1), \tag{2.19}$$

that is, that (2.16) holds true.

By letting

$$\beta = \gamma - 1 = 0$$

in Theorem 2, we arrive at

**Corollary 2.** *If the function $f \in \mathcal{A}$ satisfies the inequality:*

$$|zf''(z)| < \frac{1-\alpha}{4} \quad (z \in \mathcal{U};\ 0 \leqq \alpha < 1), \tag{2.20}$$

*then*

$$\Re\{f'(z)\} > \frac{1+\alpha}{2} \quad (z \in \mathcal{U};\ 0 \leqq \alpha < 1). \tag{2.21}$$

**Remark 2.** An analogous result (which apparently is *not* contained in Corollary 2) was proven earlier by Singh and Singh [5, p. 310, Corollary 1], which asserted that, if the function $f \in \mathcal{A}$ satisfies the inequality:

$$\left|zf''(z)\right| < 1 \quad (z \in \mathcal{U}),$$

then $f \in \mathcal{C}$.

## 3. Starlikeness and Convexity

In this section, we first prove the following result (Theorem 4 below), which involves the already introduced principle of subordination between analytic functions (see Section 1).

**Theorem 4.** *If the function $f \in \mathcal{A}$ satisfies the inequality:*

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) < \begin{cases} \dfrac{5\lambda-1}{2(\lambda+1)} & (z \in \mathcal{U};\ 1<\lambda \leqq 2) \\[2ex] \dfrac{\lambda+1}{2(\lambda-1)} & (z \in \mathcal{U};\ 2<\lambda<3) \end{cases} \tag{3.1}$$

*for some $\lambda$ $(1<\lambda<3)$, then*

$$\frac{zf'(z)}{f(z)} \prec \frac{\lambda(1-z)}{\lambda-z}. \tag{3.2}$$

*The result is sharp for the function $f$ given by*

$$f(z) = z\left(1-\frac{z}{\lambda}\right)^{\lambda-1}. \tag{3.3}$$

**Proof.** Let us define the function $w$ by

$$\frac{zf'(z)}{f(z)} = \frac{z[1-w(z)]}{\lambda-w(z)} \quad (w(z) \neq \lambda;\ z \in \mathcal{U};\ 1<\lambda<3). \tag{3.4}$$

Then, clearly, $w$ is analytic in $\mathcal{U}$ *with* $w(0)=0$. By logarithmic differentiation of both sides of (3.4), we also find that

$$1+\frac{zf''(z)}{f'(z)} = \frac{\lambda[1-w(z)]}{\lambda-w(z)} - \frac{zw'(z)}{1-w(z)} + \frac{zw'(z)}{\lambda-w(z)} \quad (z \in \mathcal{U}). \tag{3.5}$$

Assuming now that there exists a point $z_0 \in \mathcal{U}$ such that

$$|w(z_0)| = 1 \quad \text{and} \quad |w(z)| < 1 \quad \text{when} \quad |z| < |z_0|,$$

if we apply Lemma 1 just as we did in the proof of Theorem 1, we shall obtain

$$\begin{aligned}
&\Re\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right)\\
&=\Re\left(\frac{\lambda\left(1-e^{i\theta}\right)}{\lambda-e^{i\theta}}\right)-\Re\left(\frac{ce^{i\theta}}{1-e^{i\theta}}\right)+\Re\left(\frac{ce^{i\theta}}{\lambda-e^{i\theta}}\right)\\
&=\frac{\lambda(\lambda+1)(1-\cos\theta)}{1+\lambda^2-2\lambda\cos\theta}+\frac{c}{2}+\frac{c(\lambda\cos\theta-1)}{1+\lambda^2-2\lambda\cos\theta}\\
&=\frac{\lambda+1}{2}+\frac{\left(\lambda^2-1\right)(c+1-\lambda)}{2\left(1+\lambda^2-2\lambda\cos\theta\right)}\\
&\geqq\frac{\lambda+1}{2}+\frac{\left(\lambda^2-1\right)(2-\lambda)}{2\left(1+\lambda^2-2\lambda\cos\theta\right)}\qquad\left(z_0\in\mathcal{U};\ 1<\lambda<3\right),
\end{aligned}$$

which yields the inequality:

$$\Re\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right)\geqq\begin{cases}\dfrac{5\lambda-1}{2(\lambda+1)} & \left(z_0\in\mathcal{U};\ 1<\lambda\leqq 2\right)\\[2ex] \dfrac{\lambda+1}{2(\lambda-1)} & \left(z_0\in\mathcal{U};\ 2<\lambda<3\right).\end{cases}\tag{3.6}$$

Since (3.6) obviously contradicts our hypothesis (3.1), we conclude that

$$|w(z)|<1\quad(z\in\mathcal{U}),$$

that is, that

$$\left|\frac{zf'(z)}{f(z)}-\frac{\lambda}{\lambda+1}\right|<\frac{\lambda}{\lambda+1}\quad(z\in\mathcal{U};\ 1<\lambda<3),\tag{3.7}$$

which implies the subordination (3.2) asserted by Theorem 4.

Finally, for the function $f$ given by (3.3), we have

$$\frac{zf'(z)}{f(z)}=\frac{\lambda(1-z)}{\lambda-z},\tag{3.8}$$

which evidently completes our proof of Theorem 4.

**Remark 3.** A special case of Theorem 4 when $\lambda=2$ was given earlier by Singh and Singh [5, p. 313, Theorem 6].

Lastly, since

$$f(z)\in\mathcal{K}(\alpha)\Longleftrightarrow zf'(z)\in\mathcal{S}^*(\alpha)\quad(0\leqq\alpha<1),\tag{3.9}$$

whose special case, when $\alpha=0$, is the familiar Alexander theorem (*cf.*, *e.g.*, Duren [1, p. 43, Theorem 2.12]), Theorem 4 can be applied in order to deduce

**Corollary 3.** *If the function $f \in \mathcal{A}$ satisfies the inequality:*

$$\Re\left(\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right) < \begin{cases} \dfrac{3(\lambda-1)}{2(\lambda+1)} & (z \in \mathcal{U};\ 1 < \lambda \leqq 2) \\ \dfrac{3-\lambda}{2(\lambda-1)} & (z \in \mathcal{U};\ 2 < \lambda < 3) \end{cases} \tag{3.10}$$

*for some* $\lambda$ $(1 < \lambda < 3)$, *then*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{\lambda(1-z)}{\lambda - z} \tag{3.11}$$

*The result is sharp for the function* $f$ *given by*

$$f'(z) = \left(1 - \frac{z}{\lambda}\right)^{\lambda-1} \tag{3.12}$$

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